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Mean Field Upper Bound on the Transition Temperature in Multicomponent Ferromagnets[1]

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Using local Ward identities and a new correlation inequality, we prove that the mean field transition temperature is an upper bound on the true transition temperature in multicomponent Heisenberg-type classical ferromagnets.

In this paper we consider D-component spins $\sigma_\alpha^{(i)}$, $i = 1,\dots,D$, $\alpha \in Z^\nu$, forced to lie on the unit sphere S^{D-1}, i.e.,

$$\sum_{i=1}^{D} (\sigma_\alpha^{(i)})^2 = 1$$

Given a translation-invariant function $J(\alpha - \beta)$ on $Z^\nu \times Z^\nu$, we consider for $\Lambda \subset Z^\nu$ the Hamiltonian

$$H_\Lambda = - \sum_{\alpha,\beta\in\Lambda} J(\alpha - \beta) \sum_{i=1}^{D} \sigma_\alpha^{(i)}\sigma_\beta^{(i)} - h \sum_{\alpha\in\Lambda} \sigma_\alpha^{(1)}$$

and form the partition function with a priori measure the usual one on S^{D-1}. The spontaneous magnetization is as usual the right-hand derivative of the pressure $\lim_{h\downarrow 0}[p(h) - p(0)]/h$ at $h = 0$. Here we will prove:

Theorem. Let $D \geqslant 2$. If $J(\alpha - \beta) \geqslant 0$ and

$$\mathscr{J} \equiv \sum_{\alpha} J(\alpha) < D \tag{1}$$

then the spontaneous magnetization is zero.

Remarks 1. $\mathscr{J} = D$ is the mean field value. It is probably the best value for which the theorem holds, since using either the methods of Ref. 9 or Ref. 6,

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it should be possible to construct models with $\mathscr{J} = D + \epsilon$ and spontaneous magnetization.

2. This is one of a growing number of results that mean field transition temperatures are upper bounds on the true transition temperature. For general Ising models, this is proven in Ref. 10, and recently it has been shown[2] how to use Dobrushin's improved uniqueness theorem[3] to prove such a result for very general one-component models.

3. That $|\boldsymbol{\sigma}_\alpha| = 1$ is not really important. The same proof works if the spherical measure is replaced by an arbitrary, rotation-invariant measure $d\mu$ on R^D with compact support so long as $\mathscr{J}\sup\{|\sigma|^2 | \sigma \in \operatorname{supp}(d\mu)\} < D$. However, it is only for the spherical case that the result is mean field and therefore only in that case that it is presumably the best possible.

4. For $D = 1$, the result follows from one of Griffiths.[8] For $D > 1$, Dyson *et al.*,[5] using results of Brascamp and Lieb,[1] proved the result with D replaced by $\frac{1}{2}D$. For $D = 2, 3, 4$ the result was proven by Driessler *et al.*,[4] who proved the result for $D \geqslant 5$ with D replaced by $(D - 1)$. The restriction to $D \leqslant 4$ came from the use of correlation inequalities. Our proof here is slightly more miserly in the use of such inequalities and depends on one inequality we prove below. Given this inequality, we follow the proof of Ref. 4 nearly exactly.

Lemma. For any ferromagnetic model of the above type

$$\langle\{(\sigma_\alpha^{(1)})^2 - (\sigma_\alpha^{(2)})^2\}\sigma_\beta^{(1)}\rangle \geqslant 0$$

in the region $h \geqslant 0$, all α, β.

Remark. The proof extends to show positivity of

$$\left\langle \prod_\alpha \{[\sigma_\alpha^{(1)}]^{k_\alpha} - [\sigma_\alpha^{(2)}]^{k_\alpha}\}(\sigma_\alpha^{(1)})^{l_\alpha} \right\rangle$$

for any positive, integral k_α, l_α.

Proof. Change variables to

$$\boldsymbol{\sigma}_\alpha = (r_\alpha \cos\theta_\alpha, r_\alpha \sin\theta_\alpha, \sigma_\alpha^{(3)},\ldots,\sigma_\alpha^{(D)})$$

Expanding H in the numerator of the Gibbs state and using

$$\cos^2\theta - \sin^2\theta = \cos(2\theta), \qquad \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2)$$

we see that it suffices to show that for the uncoupled expectation $\langle\cdots\rangle_0$

$$\left\langle \prod_\alpha (\cos 2\theta_\alpha)^{a_\alpha} r^{b_\alpha} (\cos\theta_\alpha)^{c_\alpha} (\sigma_\alpha^{(3)})^{d_\alpha(3)} \cdots (\sigma_\alpha^{(D)})^{d_\alpha(D)} \times \prod_{\alpha<\beta} \cos(\theta_\alpha - \theta_\beta)^{e(\alpha,\beta)} \right\rangle_0 \geqslant 0 \tag{2}$$

Since,[7] using positive-definiteness arguments,

$$\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_m \prod_\mu \cos(n_{\mu 1}\theta_1 + \cdots + n_{\mu m}\theta_m) \geqslant 0$$

the result (2) is known. ■

Proof of the Theorem. Computing expectation values with periodic boundary conditions and $h \neq 0$, we have as in Ref. 3 the Ward identity

$$\begin{aligned}\langle \sigma_\alpha^{(1)} \rangle &\leqslant \sum_\beta J(\alpha - \beta) \langle \alpha_\alpha^{(2)} [\sigma_\alpha^{(2)} \sigma_\beta^{(1)} - \sigma_\alpha^{(1)} \sigma_\beta^{(2)}] \rangle + h \langle (\sigma_\alpha^{(2)})^2 \rangle \\ &\leqslant \sum_\beta J(\alpha - \beta) \langle (\sigma_\alpha^{(2)})^2 \sigma_\beta^{(1)} \rangle + h\end{aligned}$$

since $\langle \sigma_\alpha^{(1)} \sigma_\alpha^{(2)} \sigma_\beta^{(2)} \rangle \geqslant 0$ and $\langle (\sigma_\alpha^{(2)})^2 \rangle \leqslant \langle |\sigma|^2 \rangle = 1$. By symmetry and the lemma

$$\begin{aligned}D \langle (\sigma_\alpha^{(2)})^2 \sigma_\beta^{(1)} \rangle &= \langle (\sigma_\alpha^{(2)})^2 \sigma_\beta^{(1)} \rangle + \sum_{j=2}^{D} \langle (\sigma_\alpha^{(j)})^2 \sigma_\beta^{(1)} \rangle \\ &\leqslant \sum_{j=1}^{D} \langle (\sigma_\alpha^{(j)})^2 \sigma_\beta^{(1)} \rangle = \langle \sigma_\beta^{(1)} \rangle\end{aligned}$$

Taking the volume to infinity and letting $m(h)$ be the magnetization, we see that

$$m(h) \leqslant D^{-1} \mathscr{J} m(h) + h$$

so, if $\mathscr{J} < D$, $m(h) \downarrow 0$ as $h \downarrow 0$. ■

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